1 ANALYSIS

1.1 ASAP and the Rudiments of Gaussian Beam Decomposition

The Retro-reflection analysis performed on the laser train of the EOTS uses a program distributed by the Breault Research Organization (BRO). ASAP is different from other optical programs in that the optical components are entered in as physical entities, and ray tracing is nonsequential. A feature of this program different from other optical programs such as Code V and Zemax, is light is paraxially propagated using a parabasal ray, and the field reconstructed via Gaussian Decomposition.

Code V's method for calculating diffraction images in an optical system requires that a wave front be constructed using the optical path differences (OPD's) of hundreds of rays. This method is accomplished by interpolating the wave front using the OPD's at the exit pupil and either Fourier Transforming or using the vector diffraction kernel to get the resultant phase and amplitude of the field.

Since the electromagnetic field is linear, it obeys the principle of superposition, wherein an arbitrary field incident on a system can be decomposed into a set of elementary fields. The concept of decomposition, propagation, and recombination is what is done in the angular spectrum technique. The field incident on a system is decomposed into a set of plane waves, (via the Fourier Transform), propagated, using a transfer function, and then recombined (via the Inverse Fourier Transform). A predominant problem with this technique is that when the field is under sampled below the Nyquist Frequency, aliasing occurs. In Gaussian decomposition, the field is decomposed into a set of Gaussian fields and the parabasal ray is easily propagated through the system using a paraxial transfer function. The fields of the set of parabasal rays are then simply recombined using superpositioning of the fields. A benefit of this method is that Gaussian beams, when propagated through an optical system, do not change their mathematical form. Thus, when recombining the propagated fields, the problems of aliasing are eliminated. One draw back of this technique is that if the parabasal ray deviates from the paraxial ray (i.e. becomes highly divergent) then inaccurate results can occur, requiring the resultant parabasal ray grid to be decomposed into a set of rays of finer resolution and lower divergence.

Derivation of the Parabasal Ray

The derivation of the parabasal ray is taken directly out of Verdeyen[1], where the wave equation given in equation (1) is sighted as the fundamental equation for propagation of Gaussian beams in free space, and has the same basic form as the time-dependent Schrodinger equation,

$$\nabla_t^2 \psi - 2kj \frac{\partial \psi}{\partial z} = 0 \qquad (1)$$

where the subscript (t) designates the transverse mode of the field. To keep the mathematics to a minimum, we seek a solution that is cylindrically symmetric. Therefore wave equation in equation (1) is re-expressed in cylindrical coordinate by expanding the Laplancian of the equation in terms of the radius. Thus, the form of equation (1) becomes that given in equation (2).

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) - 2kj\frac{\partial \psi}{\partial z} = 0 \quad (2)$$

The standard method in seeking a solution to the partial differential equation (PDQ) given in (2) is to solve it as a variable separable. An alternate method is to select a standard form of the solution and then force the unknown coefficients or functions to fit the equation. Thus the standard form for a field propagating in free space is given in equation (3),

$$\psi_0(P(z), q(z)) = e^{-j(P(z)}\exp(-j(kr^2/2q(z)) \quad (3)$$

where the subscript (0) indicates the fundamental form of the transverse field. The objective in seeking a solution to the partial differential equation of this type, is to reduce the partial differential into a set ordinary differential equations (ODE) for the unknown functions P(z) and q(z).

Thus, after substituting all appropriate derivatives of the function $\psi_0(P(z), q(z))$ into the PDE and grouping the terms of (r) together, we get the expression given in equation (4)

$$\left\{\left[\left(\frac{k^2}{q^2(z)}\right)\left(\frac{\partial q}{\partial z} - 1\right)\right]r^2 - 2k\left[\frac{\partial P}{\partial z} + \frac{j}{q(z)}\right]r^0\right\}\psi_0 = 0 \quad (4)$$

For the assumed form of $\psi_0(P(z), q(z))$ to be a solution, every factor containing a power of (r) must vanish or become equal to zero. This yields the two following ODE's given in equations (5) and (6).

$$\frac{dq}{dz} = 1 \quad (5)$$

$$\frac{dP}{dz} = \frac{-j}{q(z)} \quad (6)$$

The two ODE's are decoupled from each other, and can be solved as two independent ODE's. The solution to the first differential equation if solved as a variable separable yields the trivial solution given in equation (7),

$$q(z) = z + q_0 \quad (7)$$

where q_0 is the value of $q(z)$ at z=0.

Assume q(z) to be complex, this implies that z is real, and any real part in the way q_0 only constitutes a shift in the spatial coordinates. To absorb the real part of the equation we simply start the coordinate at z=0 and let q_0 be imaginary (i.e. $q_0 = jz_0$). Rewriting equation (7) in terms of z yields the expression given in equation (8).

$$q(z) = z + jz_0 \quad (8)$$

If equation (8) is substituted into equation (3) and letting z=0, we get the expression of ψ_0 at z=0 yielding the results given in equation (9).

$$\psi_0(0) = \exp\left(\frac{-jkr^2}{j2z_0}\right)\exp(-jP(0)) = \exp\left(\frac{-kr^2}{2z_0}\right)e^{-jP(0)} \quad (9)$$

For the value of $r = \sqrt{2z_0/k}$ the amplitude of the field falls from 1 to 1/e. This quantity is a scale factor for the transverse radius of the field and is denoted as the variable (w_0), if we substituted the value of (w_0) in place of (r) we find an expression relating (z_0) to (w_0) which is given in equation (10)

$$w_0 = \sqrt{\frac{2z_0}{k}} = \sqrt{\frac{\lambda z_0}{\pi}} \Rightarrow z_0 = \frac{\pi w_0^{\ 2}}{\pi} \quad (10)$$

At any point along the propagation axis, the value of (q) changes according to $q(z) = z + q_0$. The reciprocal of the q(z) is of interest. Therefore we examine 1/ q(z) which is expressed in equation (11).

$$\frac{1}{q(z)} = \frac{1}{z + jz_0} = \frac{z}{z^2 + z_0^{\ 2}} - \frac{jz_0}{z^2 + z_0^{\ 2}} \qquad (11)$$

Substitution of equation (11) into equation (3) yields $\psi_0(P(z), z + z_0)$ as given in equation (12).

$$\psi_0(P(z), z + z_0) = \exp\left(\frac{-kz_0r^2}{2\left(z^2 + z_0^{\ 2}\right)}\right)\exp\left(\frac{-jkzr^2}{2\left(z^2 + z_0^{\ 2}\right)}\right)e^{-jP(z)} \quad (12)$$

We realize that the term multiplying r^2 in the first exponential factor given in equation (12) is a measure of the spread of the beam, which is now expressed as a function of (z). Therefore we obtain the function w(z) for the spread of the beam as a function of (z) given in equation (13).

$$w^2(z) = \frac{2}{kz_0}\left(z^2 + z_0{}^2\right) = \frac{2z_0}{k}\left[1 + \left(\frac{z}{z_0}\right)^2\right] = w_0\left(1 + \left(\frac{\lambda z}{\pi w_0{}^2}\right)^2\right) \qquad (13)$$

The term multiplying r^2 in the second exponential factor given in equation (12) is a measure of the radius of curvature of the field expressed as a function of z. Thus we get the expression for R(z) given in equation (14).

$$R(z) = \frac{z^2 + z_0{}^2}{z} = z\left(1 + \left(\frac{z_0}{z}\right)^2\right) = z\left(1 + \left(\frac{\pi w_0{}^2}{\lambda z}\right)^2\right) \qquad (14)$$

Thus we have all the key mathematical components for defining the parabasal ray, which are shown in Figure 1. The function $w^2(z)$ defines the size of the waist of the parabasal ray propagating in free space as a function of (z), and $R(z)$ defines the curvature of the field at as a function of (z).

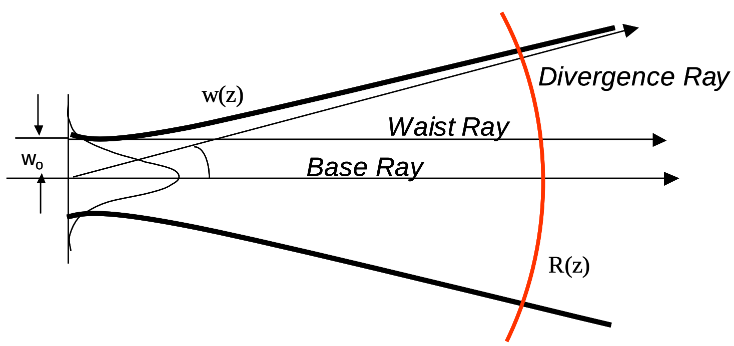

Figure 1 – Parabasal Ray Set

Gaussian Decomposition

To illustrate the principles of Gaussian decomposition, a Mathematical Analog of reconstruction of the field for the plane wave diffraction through a slit is presented. The standard Linear Systems Analysis of slit diffraction (via the Fourier Transform) is compared to a slit composed of a set of Gaussian fields.

In the following example we choose a slit of unit height and unit width. The slit is mathematically represented with the rectangle function of unit amplitude and width, given in equation (15).

$$\Pi(x) = \begin{cases} 1 & when.\ |x| \le 1/2 \\ 0 & |x| > 1/2 \end{cases} \qquad (15)$$

In order to get the amplitude of the field we apply the Forward Fourier Transform $\Im\{\Pi(x)\}$ which is given in equation (16).

$$\Im\{\Pi(x)\} = \int_{-\infty}^{\infty} \Pi(x)\exp(-2\pi j\nu x)dx = \int_{-1/2}^{1/2} \exp(-2\pi j\nu x)dx = \int_{-1/2}^{1/2} (\cos(2\pi\nu x) + j\sin(2\pi\nu x))dx$$

(16)

Using the real part of the last integral yields equation (17).

$$\Re\left(\int_{-1/2}^{1/2} (\cos(2\pi\nu x) + j\sin(2\pi\nu x))dx\right) = \int_{-1/2}^{1/2} \cos(2\pi\nu x)dx = \frac{\sin(\pi\nu)}{\pi\nu} \qquad (17)$$

In applying Gaussian decomposition to the slit, a set of super positioned Gaussian functions denote as g(x) are added together to form the slit function specified by the boundary conditions predicated by $\Pi(x)$. We define the Gaussian field function in equation (18),

$$g(x) = e^{-\left(\frac{x}{w_0}\right)^2} \qquad (18)$$

where, w_0 is radius at $1/e$ of the amplitude of the field.

In order to apply the superpositioning to $g(x)$ we define $comb(x/s)$, which is a set of delta functions equally spaced by (s). We then multiply the comb function by $\Pi(x)$ in order to restrict the function over the boundary conditions predicated by the slit. The product of these two equations is defined in equation (19),

$$comb_{slit}(x/s) = \Pi(x)comb(x/s) = \sum_{k=-n/2}^{n/2} \delta(x - ks) \qquad (19)$$

where $n \in \{2,4,...2m\}$ is the number of sampled points, and s=1/n

The composed slit function is constructed by the convolution of g(x) and $comb_{slit}(x/s)$ given in equation (20)

$$SL(x) = \int_{-\infty}^{\infty} g(x-\xi)comb_{sit}(\xi / s)d\xi = \sum_{k=-n/2}^{n/2} e^{-\left[\frac{x-ks}{w_0}\right]^2} \quad (20)$$

Figure 2 shows the synthetically composed Gaussian slit function along with $\Pi(x)$. The slit function is the result of the superpositioning of several Gaussian fields separated by the distance (s).

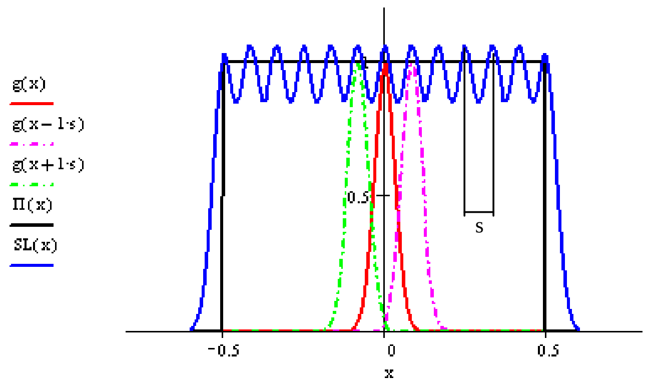

Figure 2 – Gaussian Composite Slit Function

To get the amplitude of the resulting diffraction pattern, we again apply the Forward Fourier Transform on the slit function $SL(x)$ which yields equation (21).

$$\Im[SL(x)] = \int_{-\infty}^{\infty} SL(x)e^{-2\pi j vx}dx = \sum_{k=-n/2}^{n/2} \int_{-\infty}^{\infty} e^{-\left[\frac{x-ks}{w_0}\right]^2} e^{-2\pi j vx}dx \quad (21)$$

By adding the two exponents given in equation (21) together and expanding their terms we get the following relationship given in equation (22).

$$\sum_{k=-n/2}^{n/2} \int_{-\infty}^{\infty} e^{-\left[\frac{x-ks}{w_0}\right]^2} e^{-2\pi j vx}dx = \sum_{k=-n/2}^{n/2} \int_{-\infty}^{\infty} e^{-\left[\frac{x^2}{w_0^2} - \frac{2x\left(kx+\pi w_0^2 vj\right)}{w_0^2} + \frac{(ks)^2}{w_0^2}\right]^2} dx \quad (22)$$

By applying the integral identity given (23) to the right side of equation (22) yields equation (24)

$$\int_{-\infty}^{\infty} e^{-\left(a\cdot x^2+b\cdot x+c\right)}\,dx = \sqrt{\frac{\pi}{a}}\cdot e^{\frac{-\left(b^2-4\cdot a\cdot c\right)}{4\cdot a}} \qquad (23)$$

$$\Im[SL(x)] = \sqrt{\pi}\,w_0 \sum_{k=n/2}^{n/2} \exp[-(\pi w_0 \nu)]^2 \exp[-(2\pi j s\,\nu)] \qquad (24)$$

By expanding the imaginary component of the exponent into its Euler components and only using the real part of the expression we arrive at our final solution, given in equation (25)

$$\Im[SL(x)] = \sqrt{\pi}\,w_0 \exp\left[-(\pi w_0 \nu)^2\right] \sum_{k=-\frac{n}{2}}^{\frac{n}{2}} \cos(2\pi k s\,\nu) \qquad (25)$$

In Figure 3 sinc($\pi\upsilon$) is compared to $\Im[SL(x)]$. The slit function *SL*(x) was constructed with a set of 32 superposition Gaussian fields, separated by parameter s=0.031 and a waist w_0=0.017

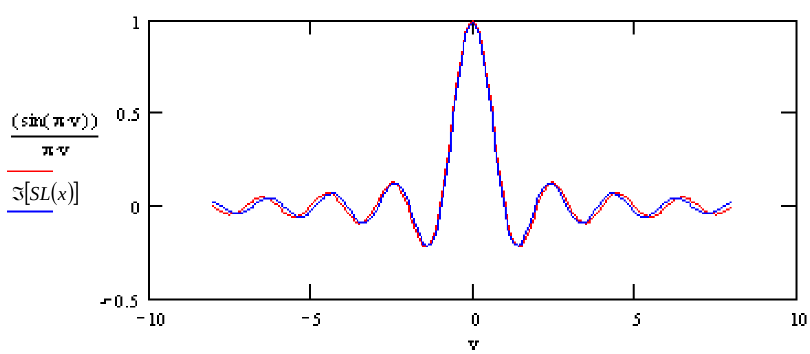

Figure 3 – Comparison of sinc($\pi\upsilon$) and $\Im[SL(x)]$

The preceding mathematical analog demonstrates the principles of Gaussian decomposition and reconstruction, but should not be confused with the internal kernel used by ASAP, marketed by BRO or FRED marketed by Photon Engineering, which are stray light analysis programs specifically designed to propagate fields via Gaussian decomposition. These programs propagate Gaussian beams by tracing the components of the parabasal ray geometrically and paraxially. The base ray of the parabasal bundle is traced exactly. The waist rays and divergence rays are traced using a paraxial transfer function. By knowing the position of the waist rays and divergence rays relative to the base ray, the corresponding amplitude and phase for the Gaussian beam is reconstructed. Therefore, for a grid of parabasal rays traced though an optical system, the phase and amplitude of the field exiting an optical system is not recombined using the Fourier Transform, but is recombined using the kernel intrinsic to the program that reconstructs the optical field which has taken into account all aberrations and diffraction effects of the optical system, and the Wavefront of the field is composed using Huygens principle for the superpositioning of spherical wavefronts.